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Positive curvature property for some hypoelliptic heat kernels

Bin Qian *

Abstract

In this note, we look at some hypoelliptic operators arising from nilpotent rank 2 Lie algebras. In particular, we concentrate on the diffusion generated by three Brownian motions and their three Lévy areas, which is the simplest extension of the Laplacian on the Heisenberg group \mathbb{H} . In order to study contraction properties of the heat kernel, we show that, as in the case of the Heisenberg group, the restriction of the sub-Laplace operator acting on radial functions (which are defined in some precise way in the core of the paper) satisfies a non negative Ricci curvature condition (more precisely a $CD(0, \infty)$ inequality), whereas the operator itself does not satisfy any $CD(r, \infty)$ inequality. From this we may deduce some useful, sharp gradient bounds for the associated heat kernel.

Keywords: Γ_2 curvature, Heat kernel, Gradient estimates, Sublaplacian, Three Brownian motions model.

2000 MR Subject Classification: 58J35 43A80

1 Introduction

In the study of the long (or small) time behavior (e.g. gradient estimates, ergodicity etc.) of simple linear parabolic evolution equations, one often uses lower bounds on the Ricci curvature associated to the generator of the heat kernel, see for example [1, 10, 17] and the references therein. But this method fails in general in hypoelliptic evolution equations, since the Ricci (Γ_2 -) curvature in even the simplest example of the Heisenberg group can not be bounded below as explained e.g. in [9, 2]. Nevertheless, in the Heisenberg group case, many properties of the elliptic case remain true, and we shall details later some of the most interesting ones.

Let us recall first some basic facts.

The elliptic case

Let M be a complete Riemannian manifold of dimension n and let $\mathcal{L} := \Delta + \nabla h$, where Δ is the Laplace-Beltrami operator. For $t \geq 0$, denote by P_t the heat semigroup generated

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by \mathcal{L} (that is formally $P_t = \exp(t\mathcal{L})$). For smooth enough function f, g , one defines (see [1])

$$\begin{aligned}\Gamma(f, g) &= |\nabla f|^2 = \frac{1}{2}(\mathcal{L}fg - f\mathcal{L}g - g\mathcal{L}f), \\ \Gamma_2(f, f) &= \frac{1}{2}(\mathcal{L}\Gamma(f, f) - 2\Gamma(f, \mathcal{L}f)) = |\nabla \nabla f|^2 + (\text{Ric} - \nabla \nabla h)(\nabla f, \nabla f).\end{aligned}$$

We have the following well-known proposition, see Proposition 3.3 in [1].

Proposition A. For every real $\rho \in \mathbb{R}$, the following are equivalent

- (i). $CD(\rho, \infty)$ holds. That is $\Gamma_2(f, f) \geq \rho\Gamma(f, f)$.
- (ii). For $t \geq 0$, $\Gamma(P_t f, P_t f) \leq e^{-2\rho t} \Gamma(f, f)$.
- (iii). For $t \geq 0$, $\Gamma(P_t f, P_t f)^{\frac{1}{2}} \leq e^{-\rho t} \Gamma(f, f)^{\frac{1}{2}}$.

Moreover, in [7], Engoulatov obtained the following gradient estimates for the heat kernels in Riemannian manifolds.

Theorem B. Let M be a complete Riemannian of dimension n with Ricci curvature bounded from below, $\text{Ric}(M) \geq -\rho$, $\rho \geq 0$.

- (i). Suppose a non-collapsing condition is satisfies on M , namely, there exist $t_0 > 0$, and $\nu_0 > 0$, such that for any $x \in M$, the volume of the geodesic ball of radius t_0 centered at x is not too small, $\text{Vol}(B_x(t_0)) \geq \nu_0$. Then there exist two constants $C(\rho, n, \nu_0, t_0)$ and $\bar{C}(t_0) > 0$, such that

$$|\nabla \log H(t, x, y)| \leq C(\rho, n, \nu_0, t_0) \left(\frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right),$$

uniformly on $(0, \bar{C}(t_0)] \times M \times M$, where $d(x, y)$ is the Riemannian distance between x and y .

- (ii). Suppose that M has a diameter bounded by D , Then there exists a constant $C(\rho, n)$ such that

$$|\nabla \log H(t, x, y)| \leq C(\rho, n) \left(\frac{D}{t} + \frac{1}{\sqrt{t}} + \rho\sqrt{t} \right),$$

uniformly on $(0, \infty) \times M \times M$.

The three-dimensional model groups

In recent year, some focus has been set on some degenerate (hypoelliptic) situations, where the methods used for the elliptic case do not apply. Among the simplest examples of such situation are the three-dimensional groups \mathfrak{G} with Lie algebra \mathfrak{g} , where there is a basis $\{X, Y, Z\}$ of \mathfrak{g} such that

$$[X, Y] = Z, [Z, Y] = \alpha Y, [Y, Z] = \alpha X, \quad (1.1)$$

where $\alpha \in \mathbb{R}$. The analysis reduces mainly to the three cases $\alpha = 0, \alpha = 1, \alpha = -1$.

Example 1.1 (Heisenberg group, $\alpha = 0$). *The Heisenberg group can be seen the Euclidean space \mathbb{R}^3 with a group structure \circ , which is defined, for $\vec{x} = (x, y, z), \vec{y} = (x', y', z') \in \mathbb{R}^3$, by*

$$\vec{x} \circ \vec{y} = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y) \right).$$

The left invariant vector fields which are given by

$$\begin{aligned} X(f) &= \lim_{\varepsilon \rightarrow 0} \frac{f(\vec{x} \circ (\varepsilon, 0, 0)) - f(\vec{x})}{\varepsilon} = \left(\partial_x - \frac{y}{2} \partial_z \right) f, \\ Y(f) &= \lim_{\varepsilon \rightarrow 0} \frac{f(\vec{x} \circ (0, \varepsilon, 0)) - f(\vec{x})}{\varepsilon} = \left(\partial_y + \frac{x}{2} \partial_z \right) f, \\ Z(f) &= \lim_{\varepsilon \rightarrow 0} \frac{f(\vec{x} \circ (0, 0, \varepsilon)) - f(\vec{x})}{\varepsilon} = \partial_z f. \end{aligned}$$

The right invariant ones are:

$$\begin{aligned} \hat{X}(f) &= \lim_{\varepsilon \rightarrow 0} \frac{f((\varepsilon, 0, 0) \circ \vec{x}) - f(\vec{x})}{\varepsilon} = \left(\partial_x + \frac{y}{2} \partial_z \right) f, \\ \hat{Y}(f) &= \lim_{\varepsilon \rightarrow 0} \frac{f((0, \varepsilon, 0) \circ \vec{x}) - f(\vec{x})}{\varepsilon} = \left(\partial_y - \frac{x}{2} \partial_z \right) f, \end{aligned}$$

The Lie algebra structure is described by the identities $[X, Y] = Z, [X, Z] = [Y, Z] = 0$. In fact, all group structures satisfying (1.1) with $\alpha = 0$ can be transformed to the case (\mathbb{R}^3, \circ) by the exponential maps, the vectors fields $\{X, Y, Z\}$ corresponding to the left ones, see Lemma 4.1 in [8], see also [5]. The natural sublaplacian operator for this model is $\mathcal{L} = X^2 + Y^2$. In this case, symmetries play an essential role : they are described by the Lie algebra of the vector fields that commute with \mathcal{L} . A basis of this Lie algebra is (\hat{X}, \hat{Y}, Z) and $\theta = x\partial_y - y\partial_x$. The last one reflects the rotational invariance of \mathcal{L} , see [2]. For this sublaplacian \mathcal{L} , we have

$$\Gamma(f, f) = (Xf)^2 + (Yf)^2,$$

and

$$\Gamma_2(f, f) = (X^2 f)^2 + (Y^2 f)^2 + \frac{1}{2}(XYf + YXf)^2 + \frac{1}{2}(Zf)^2 + 2(XZfYf - YZfXf).$$

The appearance of the mixed term $XZfYf - YZfXf$ prevents the existence of any constant $\rho \in \mathbb{R}$ such that $\Gamma_2 \geq \rho\Gamma$. Therefore the methods used in the elliptic case to prove gradient bounds cannot be used here. Nevertheless, B. Driver and T. Melcher proved in [6], the existence of a finite positive constant C_2 such that

$$\forall f \in C^\infty(\mathbb{H}, \mathbb{R}), \forall t \geq 0, \Gamma(P_t f, P_t f) \leq C_2 P_t \Gamma(f, f), \quad (1.2)$$

where P_t denotes the associated heat semigroup generated by \mathcal{L} , $C^\infty(\mathbb{H}, \mathbb{R})$ is the class of smooth function from \mathbb{H} to \mathbb{R} with all partial derivatives of polynomial growth. More recently, H. Q. Li [11] showed that there exists positive constant C_1 such that

$$\forall f \in \mathcal{P}^\infty(\mathbb{H}), \forall t \geq 0, \Gamma(P_t f, P_t f)^{\frac{1}{2}} \leq C_1 P_t (\Gamma(f, f)^{\frac{1}{2}}). \quad (1.3)$$

(See also D. Bakry et al. [2] for alternate proofs.) The gradient estimate (1.3) is much stronger than (1.2), and has many consequence in terms of functional inequalities for the heat kernel P_t , including Poincaré inequalities, Gross logarithmic Sobolev inequalities, Cheeger type inequalities, and Bobkov type inequalities, see section 6 in [2].

Let p_t be the heat kernel of P_t at 0 with respect to Lebesgue measures on \mathbb{R}^3 . In [11], H. Q. Li has also pointed out that for $t \geq 0, g \in \mathbb{H}$, there exists a positive constant C such that

$$|\nabla \log p_t|(g) \leq \frac{Cd(g)}{t}, \quad (1.4)$$

where $d(g)$ denotes the Carnot-Carhéodory distance (see (1.8)) between 0 and g . This gradient estimate is sharp and plays an important role in the proof of (1.3).

In the case $\alpha = 1$, the Lie algebra is the one of the $SU(2)$ Lie group, and this case has been studied by F. Baudoin and M. Bonnefont in [4]. They show that a modified form of (1.3) and (1.4) hold. Other generalizations of Heisenberg group are the so-called Heisenberg type group. They have been studied by H. Q. Li in [12, 13], where he shows that (1.3) and (1.4) hold in this setting. In this note, we shall focus on a group that we may call, the three Brownian motions model. It can be seen as another typical simple example of hypoelliptic operator, but the structure is more complex than the Heisenberg (type) groups and the method of H.Q. Li fails to study the precise gradient bounds in this context.

For this model, we shall first look at the symmetries, that is characterize all the vector fields which commute with the sublaplacian operator \mathcal{L} , see Proposition 2.1. The infinitesimal rotations are those vector fields which vanish at 0 and a radial function is a function which vanishes on infinitesimal rotations. In this case, although the Ricci curvature is everywhere $-\infty$, refer to [9, 2], we shall prove that the Γ_2 curvature is still positive along the radial directions, as it is the case for the Heisenberg group, see Proposition 3.1. As a consequence, the same form of gradient estimate (1.4) holds by combining the method developed by F. Baudoin and M. Bonnefont in [4] with the method in [12]. It is worth recalling that in [3], D. Bakry et al. have obtained the Li-Yau type gradient estimates for the three dimensional model group by applying Γ_2 -techniques. In our setting, it is easy to see that this type of gradient estimate also holds.

The three Brownian motions model

The three Brownian motions model $\mathfrak{N}_{3,2}$, see section 4 in [8], can be described as the Euclidean space \mathbb{R}^6 with a the following group structure \circ , which is defined by for $\vec{x} = (x_1, x_2, x_3, y_1, y_2, y_3)$, $\vec{y} = (x'_1, x'_2, x'_3, y'_1, y'_2, y'_3) \in \mathbb{R}^6$,

$$(x_1, x_2, x_3, y_1, y_2, y_3) \circ (x'_1, x'_2, x'_3, y'_1, y'_2, y'_3) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3, y_1 + y'_1 + \frac{1}{2}(x_2 x'_3 - x_3 x'_2), \\ y_2 + y'_2 + \frac{1}{2}(x_3 x'_1 - x_1 x'_3), y_3 + y'_3 + \frac{1}{2}(x_1 x'_2 - x_2 x'_1)).$$

For simplification, we make the convention that the index $i \equiv j \pmod{3}$, and here we choose $j = 1, 2, 3$. In what follows, denote $\mathfrak{N}_{3,2} = (\mathbb{R}^6, \circ)$ be the three Brownian motions model. The three left invariant vector fields which are given, for $1 \leq i \leq 3$, by

$$X_i f = \lim_{\varepsilon \rightarrow 0} \frac{f(\vec{x} \circ (\varepsilon_1, \varepsilon_2, \varepsilon_3, 0, 0, 0)) - f(\vec{x})}{\varepsilon} = \left(\partial_i - \frac{x_{i+1}}{2} \hat{\partial}_{i+2} + \frac{x_{i+2}}{2} \hat{\partial}_{i+1} \right) f, \\ Y_i f = \lim_{\varepsilon \rightarrow 0} \frac{f(\vec{x} \circ (0, 0, 0, \varepsilon_1, \varepsilon_2, \varepsilon_3)) - f(\vec{x})}{\varepsilon} = \hat{\partial}_i f,$$

where $\varepsilon_i = \varepsilon$ and $\varepsilon_j = 0$ for $j \neq i$. Here we use the notation $\hat{\partial}_i = \partial_{y_i}$.

The right invariant vector fields which are gives \hat{X}_i , for $1 \leq i \leq 3$, $\varepsilon_i = \varepsilon$ and $\varepsilon_j = 0$ for $j \neq i$,

$$\hat{X}_i f = \lim_{\varepsilon \rightarrow 0} \frac{f((\varepsilon_1, \varepsilon_2, \varepsilon_3, 0, 0, 0) \circ \vec{x}) - f(\vec{x})}{\varepsilon} = \left(\partial_i + \frac{x_{i+1}}{2} \hat{\partial}_{i+2} - \frac{x_{i+2}}{2} \hat{\partial}_{i+1} \right) f.$$

There are no \hat{Y}_i 's since in this setting the left and right multiplications coincide. The Lie algebra structure is described by the formulae, for $1 \leq i, j \leq 3$,

$$[X_i, X_{i+1}] = Y_{i+2}, \quad [X_i, Y_j] = 0. \quad (1.5)$$

Similarly for all group structure satisfying (1.5) can be transformed to the case (\mathbb{R}^6, \circ) via the exponential maps, the vectors fields are corresponding to the left ones.

In what follows, we are interested in the natural sublaplacian for this model, which is defined by

$$\mathcal{L} = \sum_{i=1}^3 X_i^2.$$

The reason why we call it the three Brownian motions model is that $\frac{1}{2}\mathcal{L}$ is the infinitesimal generator of the Markov process $(\{B_i\}_{1 \leq i \leq 3}, \{\frac{1}{2} \int_0^t B_i dB_{i+1} - B_{i+1} dB_i\}_{1 \leq i \leq 3})$, where $\{B_i\}_{1 \leq i \leq 3}$ are three real standard independent Brownian motions.

For all $t \geq 0$, $P_t := e^{t\mathcal{L}}$ denotes the associated heat semigroup generated by \mathcal{L} , p_t the heat kernel of P_t at 0 with respect to Lebesgue measures on \mathbb{R}^6 . For this operator \mathcal{L} , we have

$$\Gamma(f, g) = \sum_{i=1}^3 X_i f X_i g$$

and

$$\Gamma_2(f, f) = \sum_{i,j=1}^3 (X_i X_j f)^2 - 2 \sum_{i=1}^3 X_i f (X_{i+1} Y_{i+2} f - Y_{i+1} X_{i+2} f).$$

Here again the mixed term $\sum_{i=1}^3 X_i f (X_{i+1} Y_{i+2} f - Y_{i+1} X_{i+2} f)$ prevents the existence of any constant ρ such that the curvature dimensional condition $CD(\rho, \infty)$ holds. Nevertheless, we have the following Driver-Melcher inequality, see [15],

$$\Gamma(P_t f, P_t f) \leq C P_t(\Gamma(f, f)),$$

for some positive constant C . The constant C here can be expressed explicitly following the method in [2]. Also the optimal reverse local Poincaré inequality holds, see Remark 3.3 in [2]. That is, for any $t \geq 0$ and any $f \in C_c^\infty(\mathfrak{N}_{3,2})$,

$$t\Gamma(P_t f, P_t f) \leq \frac{3}{2} (P_t(f^2) - (P_t f)^2).$$

For the H. Q. Li inequality (1.3), the methods deeply rely on the precise estimates on the heat kernel p_t and its differentials (see [11, 2]). Up to the author's knowledge, these precise estimates are not known in the three Brownian motions model, neither the H. Q. Li inequality. Nevertheless, we shall prove that one of the key gradient estimates (1.4) holds, which would be a first step for the proof of the H. Q. Li inequality in this context, see Proposition 4.2.

The dilation operator in this model is defined by $\mathcal{D} := \frac{1}{2} \sum_i x_i \partial_i + \sum_{i=1}^3 y_i \hat{\partial}_i$, and it satisfies

$$[\mathcal{L}, \mathcal{D}] = \mathcal{L}. \tag{1.6}$$

For $t \geq 0$, let $T_t = e^{t\mathcal{D}}$ be the semigroup generated by \mathcal{D} , that is

$$T_t f(x_1, x_2, x_3, y_1, y_2, y_3) = f\left(\exp\left(\frac{t}{2}\right)x_1, \exp\left(\frac{t}{2}\right)x_2, \exp\left(\frac{t}{2}\right)x_3, \exp(t)y_1, \exp(t)y_2, \exp(t)y_3\right).$$

From the commutaton relation (1.6), one deduces, for $t, s \geq 0$,

$$P_t T_s = T_s P_{e^s t}.$$

Since 0 is a fixed point of the dilation group T_t , it follows

$$P_t(f)(0) = P_1(T_{\log t}f)(0). \quad (1.7)$$

So it is enough to describe the heat kernel at any time and any point to know the operator $P_1(f)(0)$.

The natural distance, induced by the sublaplacian operator \mathcal{L} , is the Carnot-Carathéodory distance d . As usual, it can be defined from the operator \mathcal{L} only by

$$d(g_1, g_2) := \sup_{\{f: \Gamma(f) \leq 1\}} f(g_1) - f(g_2). \quad (1.8)$$

For this distance, we have the invariant and scaling properties, see [8, 17].

$$d(g_1, g_2) = d(g_2^{-1} \circ g_1, 0) := d(g_2^{-1} \circ g_1), \text{ and } d(\gamma \vec{x}, \gamma^2 \vec{y}) = \gamma d(x, y),$$

for all $g_1, g_2 \in \mathfrak{N}_{3,2}$, $\gamma \in \mathbb{R}^+$ and $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

2 Rotation vectors and Radial functions

In this section, we shall characterize all the vector fields which commute with \mathcal{L} . Obviously the right invariant vector fields $\{\hat{X}_i, Y_i\}_{i=1,2,3}$ commute with \mathcal{L} since they commute with $\{X_i\}_{i=1,2,3}$. Like the rotation vector field $\theta = x\partial_y - y\partial_x$ in the Heisenberg group, which commutes with \mathcal{L} , there are three rotation vector fields in this case

$$\theta_i = x_{i+1}\partial_{i+2} - x_{i+2}\partial_{i+1} + y_{i+1}\hat{\partial}_{i+2} - y_{i+2}\hat{\partial}_{i+1}, \quad i = 1, 2, 3.$$

It is easy to see that $\{\theta_i\}_{i=1,2,3}$ commute with \mathcal{L} and we have $[\theta_i, \theta_{i+1}] = \theta_{i+2}$, for $1 \leq i \leq 3$. We first have the

Proposition 2.1. *The vector fields which commute with \mathcal{L} are the linear combination of the following nine vector fields: the three right invariant vectors, the three rotations $\{\theta_i\}_{1 \leq i \leq 3}$, and $\hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3$, that is*

$$\mathcal{T} = \{X : X \in \text{span}\{\hat{X}_i, Y_i, 1 \leq i \leq 3\}, [\mathcal{L}, X] = 0\} = \text{Linear}\{\hat{X}_i, \theta_i, Y_i, 1 \leq i \leq 3\}. \quad (2.1)$$

Here "span" means the we consider linear combinations of the vector fields with smooth functions as coefficients, while "Linear" means the coefficients are constants.

Proof. We only need to show that the left hand side space in (2.1) is contained in the right hand side one. To this end, for any vector field $X = \sum_{i=1}^3 a_i X_i + b_i Y_i$ for some smooth function a_i, b_i , satisfies $[\mathcal{L}, X] = 0$. For $1 \leq i \leq 3$, denote $Z_i = X_i X_{i+1} + X_{i+1} X_i$, it yields $X_i X_{i+1} = \frac{Z_{i+2} + Y_{i+2}}{2}$ and $X_i X_{i+2} = \frac{Z_{i+1} - Y_{i+1}}{2}$. Notice that

$$\begin{aligned} [\mathcal{L}, X] &= \sum_{i=1}^3 \left(\mathcal{L}a_i X_i + (\mathcal{L}b_i + X_{i+1}a_{i+2} - X_{i+2}a_{i+1})Y_i + (X_{i+1}a_{i+2} + X_{i+2}a_{i+1})Z_i \right. \\ &\quad \left. + 2X_i a_i X_i^2 + 2X_i b_i X_i Y_i + 2(X_i b_{i+1} - a_{i+2})X_i Y_{i+1} + 2(X_i b_{i+2} + a_{i+1})X_i Y_{i+2} \right), \end{aligned}$$

thus we have, for $1 \leq i \leq 3$,

$$X_i a_i = X_i b_i = 0, \quad (2.2)$$

$$X_{i+1} b_{i+2} = -X_{i+2} b_{i+1} = a_i, \quad (2.2')$$

$$X_i a_{i+1} = -X_{i+1} a_i. \quad (2.2'')$$

Let us first prove the following two claims.

Claim I: For $1 \leq i \leq 3$, a_i is independent on $\{y_i, 1 \leq i \leq 3\}$ and linear in $\{x_i, 1 \leq i \leq 3\}$.

To proof the desired result, for $1 \leq i \leq 3$, we have the following commutative property: $[X_i, Y_{i+1}] = 0$, together with (2.2), it yields $X_i Y_{i+1} b_i = 0$. Since $Y_{i+1} = [X_{i+2}, X_i]$, we can get $X_i^2 X_{i+2} b_i = 0$, thus $X_i^2 a_{i+1} = 0$ by the relation $X_{i+2} b_i = a_{i+1}$. Similarly we have $X_i^2 a_{i+2} = 0$. In fine, together with $X_i a_i = 0$,

$$X_i^2 a_j = 0, \quad i, j = 1, 2, 3.$$

Since

$$\begin{aligned} [X_1, Y_3] b_2 &= 0 \Rightarrow 2X_1 X_2 a_3 = -X_2 X_3 a_1, \\ [X_3, Y_1] b_2 &= 0 \Rightarrow 2X_3 X_1 a_2 = -X_2 X_3 a_1, \\ [X_1, Y_2] b_3 &= 0 \Rightarrow 2X_1 X_2 a_3 = -X_3 X_1 a_2, \end{aligned}$$

we have

$$X_1 X_2 a_3 = X_2 X_3 a_1 = X_3 X_1 a_2 = 0.$$

Together with the fact $X_i a_j = -X_j a_i$ by (2.2''), we have $X_i X_j a_k = 0$ for i, j, k all different. Thus we can conclude

$$X_i X_j a_k = 0 \quad \text{for } 1 \leq i, j, k \leq 3. \quad (2.3)$$

Note that $Y_1 a_1 = X_2 X_3 a_1 - X_3 X_2 a_1$, $Y_2 a_1 = X_1^2 a_3$ and $Y_3 a_1 = -X_1^2 a_2$, thanks to (2.3), we get $Y_i a_1 = 0, i = 1, 2, 3$. That is a_1 is independent on $\{y_i, i = 1, 2, 3\}$. Similarly a_2, a_3 is independent of $\{y_i, i = 1, 2, 3\}$. Then from the definition X_i , we have $X_i a_j = \partial_i a_j$ for $1 \leq i, j \leq 3$. With (2.3), we can conclude that $\{a_i, i = 1, 2, 3\}$ is linear in $\{x_i, i = 1, 2, 3\}$.

Note that we can also write in the form $X = \sum_{i=1}^3 a_i \partial_i + c_i \hat{\partial}_i$, where $c_i = b_i + \frac{1}{2}(a_{i+2} x_{i+1} - a_{i+1} x_{i+2})$, $1 \leq i \leq 3$. Then we can conclude

Claim II: $\{c_i, i = 1, 2, 3\}$ is linear in $\{x_i, y_i, 1 \leq i \leq 3\}$. By Claim I, a_i is independent of y_j , together with the fact $X_i a_i = 0$, we have the equation (2.2') is equivalent to

$$\begin{aligned} \frac{1}{2} a_i &= X_{i+1} c_{i+2} + \frac{1}{2} x_{i+1} \partial_{i+1} a_i \\ &= -X_{i+2} c_{i+1} + \frac{1}{2} x_{i+2} \partial_{i+2} a_i. \end{aligned} \quad (2.4)$$

And $X_i b_i = 0$ is equivalent to

$$X_i c_i = \frac{1}{2} (x_{i+1} \partial_i a_{i+2} - x_{i+2} \partial_i a_{i+1}). \quad (2.5)$$

Using (2.2)-(2.5), the relations $[X_i, X_{i+1}] = Y_{i+2}$ and Claim I, through computation, we have

$$Y_i c_j = \partial_i a_j, \quad \text{for } i, j = 1, 2, 3. \quad (2.6)$$

Since a_i is linear in x_i , we can conclude that c_j has no second order terms in $\{x_i, y_i, 1 \leq i \leq 3\}$. By the definition of X_i and (2.5) and (2.6), we have

$$\begin{aligned} \partial_i c_i &= X_i c_i + \frac{x_{i+1}}{2} Y_{i+2} c_i - \frac{x_{i+2}}{2} Y_{i+1} c_i \\ &= \frac{1}{2} (x_{i+1} \partial_i a_{i+2} - x_{i+2} \partial_i a_{i+1}) - \frac{x_{i+1}}{2} \partial_i a_{i+2} + \frac{x_{i+2}}{2} \partial_i a_{i+1} \\ &= 0. \end{aligned}$$

By (2.4) and (2.6),

$$\begin{aligned}\partial_i c_{i+1} &= X_i c_{i+1} + \frac{x_{i+1}}{2} Y_{i+2} c_{i+1} - \frac{x_{i+2}}{2} Y_{i+1} c_{i+1} \\ &= \frac{1}{2} a_{i+2} - \frac{1}{2} x_i \partial_i a_{i+2} - \frac{1}{2} x_{i+1} \partial_{i+1} a_{i+2},\end{aligned}\tag{2.7}$$

similarly,

$$\partial_i c_{i+2} = -\frac{1}{2} a_{i+1} + \frac{1}{2} x_i \partial_i a_{i+1} + \frac{1}{2} x_{i+2} \partial_{i+2} a_{i+1}.\tag{2.8}$$

By Claim I, (2.6)-(2.8) and $\partial_i a_i = 0$, we can conclude that for $1 \leq i, j \leq 3$, $\partial_i c_j$ is constant. Thus we complete to proof Claim II.

By the above two claims and $\partial_i a_i = 0$, we can assume

$$a_i = A_{i,i+1} x_{i+1} + A_{i,i+2} x_{i+2} + B_i,$$

where $A_{i,j} = -A_{j,i}$, B_i are constants, then we have, by (2.6)-(2.8),

$$c_i = \frac{1}{2} (B_{i+1} x_{i+2} - B_{i+2} x_{i+1}) + A_{i,i+1} y_{i+1} - A_{i+2,i} y_{i+2} + D_i,$$

where D_i are constants.

If we choose $B_i = 1$ (or respectively $D_i = 1$, $A_{i,i+2} = 1$) and the other constants 0, we get $X = \hat{X}_i$ (or respectively Y_i , θ_i). Thus we complete the proof. \square

In the Heisenberg group, the radial functions f can be characterized by $\theta f = 0$. Here in our setting, as an extension of such characterization, we can give a definition of radial functions.

Definition 2.2. A smooth enough function f is called radial if and only if for $1 \leq i \leq 3$, $\theta_i f = 0$.

(Notice that here the vector fields θ_i are the commuting vector fields which vanish in 0.)

Remarks 2.3. Note that the heat kernel $(p_t)_{t \geq 0}$ is radial. The reason is that for any function f , $1 \leq i \leq 3$, $\theta_i f(0) = 0$ and $\{\theta_i\}_{1 \leq i \leq 3}$ commute with \mathcal{L} , whence they commute with the semigroup $P_t = e^{t\mathcal{L}}$. Hence, for any function f , one has $P_t \theta_i f = 0$, which, taking the adjoint of θ_i under the Lebesgue measure, which is $-\theta_i$, shows that for the density p_t of the heat kernel at 0, one has $\theta_i p_t = 0$. This explains why any information about the radial functions in turns give information on the heat kernel itself.

Remarks 2.4. For any radial function f , there exist some function g such that $f(\vec{x}, \vec{y}) = g(r_1, r_2, z)$, where $r_1 = \sum_{i=1}^3 x_i^2$, $r_2 = \sum_{i=1}^3 y_i^2$, $z = \sum_{i=1}^3 x_i y_i$. Indeed, by the definition, for $1 \leq i \leq 3$, $\theta_i f = 0$, then we have $f = f(U\vec{x}, U\vec{y})$, where U is arbitrary linear orthogonal transformation on \mathbb{R}^3 , which satisfying $U^* U = U U^* = 1$. Hence $f = \bar{f}(|\vec{x}|, |\vec{y}|, \langle \vec{x}, \vec{y} \rangle)$, for some function \bar{f} . Here is another way, by the transformation θ_i , we will directly get

$$f(\vec{x}, \vec{y}) = f(\sqrt{r_1}, 0, 0, \frac{z}{\sqrt{r_1}}, 0, \sqrt{r_2 - z^2/r_1}).$$

3 Γ_2 curvature

Recall that we can't find a constant $\rho \in \mathbb{R}$ such that $\Gamma_2 \geq \rho\Gamma$ because of the appearance of the items $X_i f X_j Y_k f$. In other words, the Ricci curvature is everywhere $-\infty$. Nevertheless we shall prove Γ_2 curvature is positive on the radial functions.

Proposition 3.1. *For any smooth radial function f , we have*

$$\Gamma_2(f, f) \geq 0.$$

Here we will give two different proofs. The first one is that we shall use directly the three equations asserting that a function is radial. Then, applying the vector fields $\{X_j\}_{1 \leq j \leq 3}$ on these equations, we get nine equations in hand. It follows that we can get the exact expressions of $\{X_i Y_j f\}_{1 \leq i, j \leq 3}$ in terms of $X_i X_j f$ and also first order terms. (In fact, we adapt the mathematical software MAPLE to do it). Then we substitute them into the formal expression of Γ_2 , and we find that Γ_2 can also be expressed in a functional non negative quadratic form.

The second way is that by the Remark 2.4, we have an expression of the sublaplacian operator acting on radial functions directly through a good parametrization, say r_1, r_2, z . Through computation, we can obtain the exact expression of Γ_2 curvature and find again that Γ_2 can be expressed in a functional non negative quadratic form, thus we are done.

The first proof. A radial function f satisfies $\theta_i f = 0$, which is equivalent to say that

$$x_{i+1} X_{i+2} f - x_{i+2} X_{i+1} f - \frac{x_{i+1}^2 + x_{i+2}^2}{2} Y_i f + \frac{x_i x_{i+1} - 2y_{i+2}}{2} Y_{i+1} f + \frac{x_i x_{i+2} + 2y_{i+1}}{2} Y_{i+2} f = 0.$$

Differentiating the above equations at the directions $\{X_j\}_{1 \leq j \leq 3}$, with the commutative relations $[X_i, X_{i+1}] = Y_{i+2}$, we get the nine differential equations, for $1 \leq i \leq 3$,

$$\begin{aligned} & x_{i+1} X_{i+2} X_i f - x_{i+2} X_{i+1} X_i f - \frac{x_{i+1}^2 + x_{i+2}^2}{2} X_i Y_i f \\ & + \frac{x_i x_{i+1} - 2y_{i+2}}{2} X_i Y_{i+1} f + \frac{x_i x_{i+2} + 2y_{i+1}}{2} X_i Y_{i+2} f = 0, \\ & X_{i+2} f + x_{i+1} X_{i+2} X_{i+1} f - x_{i+2} X_{i+1}^2 f - \frac{x_{i+1}^2 + x_{i+2}^2}{2} X_{i+1} Y_i f \\ & + \frac{x_i x_{i+1} - 2y_{i+2}}{2} X_{i+1} Y_{i+1} f + \frac{x_i x_{i+2} + 2y_{i+1}}{2} X_{i+1} Y_{i+2} f = 0, \\ & - X_{i+1} f + x_{i+1} X_{i+2}^2 f - x_{i+2} X_{i+1} X_{i+2} f - \frac{x_{i+1}^2 + x_{i+2}^2}{2} X_{i+2} Y_i f \\ & + \frac{x_i x_{i+1} - 2y_{i+2}}{2} X_{i+2} Y_{i+1} f + \frac{x_i x_{i+2} + 2y_{i+1}}{2} X_{i+2} Y_{i+2} f = 0. \end{aligned}$$

For simplicity, we will use the following notations, for $1 \leq i \leq 3$,

$$\begin{aligned} \alpha_{i+1} &:= x_{i+1} X_i f - x_i X_{i+1} f, \quad \beta_{i+1} := y_{i+1} X_i f - y_i X_{i+1} f, \\ \gamma_i &:= x_i y_{i+1} - x_{i+1} y_i, \quad \eta_i := x_i y_i + x_{i+1} y_{i+1}, \\ |x|_i^2 &:= x_i^2 + x_{i+1}^2, \quad A_i := \gamma_2 X_1 X_{i+1} f + \gamma_3 X_2 X_{i+1} f + \gamma_1 X_3 X_{i+1} f, \end{aligned}$$

and

$$|h|^2 := \sum_{i=1}^3 |h_i|^2, \quad \text{for } h = (h_1, h_2, h_3) \in \mathbb{R}^3.$$

From the above nine differential equations, we can get, for $1 \leq i \leq 3$,

$$\begin{aligned} X_i Y_{i+1} f = & -\frac{1}{2|\gamma|^2} \left((x_i x_{i+1} |x|^2 + 2y_{i+2} |x|_i^2 - 2x_{i+2} \eta_i + 4y_i y_{i+1}) \cdot (x_{i+2} X_{i+1} X_i f - x_{i+1} X_{i+2} X_i f) \right. \\ & + (x_{i+1} x_{i+2} |x|^2 - 2y_i |x|_{i+1}^2 + 2x_i \eta_{i+1} + 4y_{i+1} y_{i+2}) \cdot (x_{i+1} X_i^2 f - x_i X_{i+1} X_i f - X_{i+1} f) \\ & \left. + (x_{i+1}^2 |x|^2 + 4y_{i+1}^2) \cdot (x_i X_{i+2} X_i f - x_{i+2} X_i^2 f + X_{i+2} f) \right), \end{aligned}$$

and

$$\begin{aligned} X_i Y_{i+2} f = & -\frac{1}{2|\gamma|^2} \left((x_i x_{i+2} |x|^2 - 2y_{i+1} |x|_{i+2}^2 + 2x_{i+1} \eta_{i+2} + 4y_{i+2} y_i) \cdot (x_{i+2} X_{i+1} X_i f - x_{i+1} X_{i+2} X_i f) \right. \\ & + (x_{i+1} x_{i+2} |x|^2 + 2y_i |x|_{i+1}^2 - 2x_i \eta_{i+1} + 4y_{i+1} y_{i+2}) \cdot (x_i X_{i+2} X_i f - x_{i+2} X_i^2 f + X_{i+2} f) \\ & \left. + (x_{i+2}^2 |x|^2 + 4y_{i+2}^2) \cdot (x_{i+1} X_i^2 f - x_i X_{i+1} X_i f - X_{i+1} f) \right). \end{aligned}$$

Note that

$$\Gamma_2(f, f) = \sum_{i,j=1}^3 (X_i X_j f)^2 - 2 \sum_{i=1}^3 X_i f (X_{i+1} Y_{i+2} f - Y_{i+1} X_{i+2} f).$$

With the exact expressions of $X_i Y_j f$ in hand, through calculation, we have

$$\begin{aligned} F : &= 2|\gamma|^2 \cdot \sum_{i=1}^3 X_i f (X_{i+1} Y_{i+2} f - X_{i+2} Y_{i+1} f) \\ &= 2 \sum_{i,j=1}^3 \left(X_{i+j} f (2y_{i+j+1} \gamma_{i+1} - x_{i+j+1} y_i |x|_{i+1}^2 + x_i x_{i+j+1} \eta_{i+1}) \right. \\ &\quad \left. - X_{i+j+1} f (2y_{i+j} \gamma_{i+1} - x_{i+j} y_i |x|_{i+1}^2 + x_i x_{i+j} \eta_{i+1}) \right) X_i X_{i+j+2} f - |x|^2 \cdot |\alpha|^2 - 4|\beta|^2 \\ &= 2 \sum_{i,j=1}^3 \left(2\beta_{i+j} \gamma_{i+1} + \alpha_{i+j} \gamma_i x_{i+1} - \alpha_{i+j} \gamma_{i+2} x_{i+2} \right) X_i X_{i+j+1} f - |x|^2 \cdot |\alpha|^2 - 4|\beta|^2. \end{aligned}$$

Rearrange the items, we have

$$F = 2 \sum_{i,j=1}^3 \alpha_i x_{i+j+1} (\gamma_{i+j} X_{i+j} X_{i+1} f - \gamma_{i+j+1} X_{i+j+2} X_{i+1} f) + 4 \sum_{i=1}^3 \beta_i A_i - |x|^2 \cdot |\alpha|^2 - 4|\beta|^2.$$

Notice that

$$|\gamma|^2 \cdot \sum_{i,j=1}^3 (X_i X_j f)^2 - \sum_{i=1}^3 A_i^2 = \sum_{i,j=1}^3 (\gamma_{i+j} X_{i+j} X_{i+1} f - \gamma_{i+j+1} X_{i+j+2} X_{i+1} f)^2,$$

it follows that

$$\begin{aligned}
|\gamma|^2 \cdot \Gamma_2 &= |\gamma|^2 \cdot \sum_{i,j=1}^3 (X_i X_j f)^2 - F \\
&= \sum_{i=1}^3 (2\beta_i - A_i)^2 + \sum_{i,j=1}^3 (\gamma_{i+j} X_{i+j} X_{i+1} f - \gamma_{i+j+1} X_{i+j+2} X_{i+1} f)^2 \\
&\quad - 2 \sum_{i,j=1}^3 \alpha_i x_{i+j+1} (\gamma_{i+j} X_{i+j} X_{i+1} f - \gamma_{i+j+1} X_{i+j+2} X_{i+1} f) + |x|^2 \cdot |\alpha|^2 \\
&= \sum_{i=1}^3 (2\beta_i - A_i)^2 + \sum_{i,j=1}^3 (\gamma_{i+j} X_{i+j} X_{i+1} f - \gamma_{i+j+1} X_{i+j+2} X_{i+1} f - \alpha_i x_{i+j+1})^2.
\end{aligned}$$

Hence we complete the proof. \square

The second proof. Denote $r_1 = \sum_{i=1}^3 x_i^2$, $r_2 = \sum_{i=1}^3 y_i^2$, $z = \sum_{i=1}^3 x_i y_i$. Through calculation, we have

$$\mathcal{L}r_1 = 6, \quad \mathcal{L}r_2 = r_1, \quad \mathcal{L}z = 0,$$

and

$$\begin{aligned}
\Gamma(r_1, r_1) &= 4r_1, \quad \Gamma(r_2, r_2) = r_1 r_2 - z^2, \quad \Gamma(z, z) = r_2, \\
\Gamma(r_1, r_2) &= 0, \quad \Gamma(r_1, z) = 2z, \quad \Gamma(r_2, z) = 0.
\end{aligned}$$

For any radial functions f, g depend only on r_1, r_2, z , we have

$$\begin{aligned}
\mathcal{L}f(r_1, r_2, z) &= \partial_{r_1} f \mathcal{L}r_1 + \partial_{r_2} f \mathcal{L}r_2 + \partial_z f \mathcal{L}z + \partial_{r_1 r_1}^2 f \Gamma(r_1, r_1) + \partial_{r_2 r_2}^2 f \Gamma(r_2, r_2) + \partial_{zz}^2 f \Gamma(z, z) \\
&\quad + 2\partial_{r_1 z}^2 f \Gamma(r_1, z) + 2\partial_{r_2 z}^2 f \Gamma(r_2, z) + 2\partial_{r_1 r_2}^2 f \Gamma(r_1, r_2) \\
&= 4r_1 \partial_{r_1 r_1}^2 f + (r_1 r_2 - z^2) \partial_{r_2 r_2}^2 f + r_2 \partial_{zz}^2 f + 4z \partial_{r_1 z}^2 f + 6\partial_{r_1} f + r_1 \partial_{r_2} f \\
&:= \hat{\mathcal{L}}f,
\end{aligned}$$

where $\hat{\mathcal{L}}$ has the following expression

$$\hat{\mathcal{L}}f = 4r_1 f_{11} + (r_1 r_2 - z^2) f_{22} + r_2 f_{zz} + 4z f_{1z} + 6f_1 + r_1 f_2.$$

Hence for any radial function $f = f(r_1, r_2, z)$, $g = g(r_1, r_2, z)$,

$$\begin{aligned}
\Gamma(f, g) &:= \frac{1}{2} (\mathcal{L}(fg) - f \mathcal{L}g - g \mathcal{L}f) \\
&= \frac{1}{2} (\hat{\mathcal{L}}(fg) - f \hat{\mathcal{L}}g - g \hat{\mathcal{L}}f) \\
&= \hat{\Gamma}(f, g),
\end{aligned}$$

and also

$$\begin{aligned}
\Gamma_2(f, f) &:= \frac{1}{2} (\mathcal{L}\Gamma(f, f) - 2\Gamma(f, \mathcal{L}f)) \\
&= \frac{1}{2} (\hat{\mathcal{L}}\hat{\Gamma}(f, f) - 2\hat{\Gamma}(f, \hat{\mathcal{L}}f)) \\
&= \hat{\Gamma}_2(f, f).
\end{aligned}$$

Through direct calculation, we have

$$\begin{aligned}\Gamma(f, g) &= \hat{\Gamma}(f, g) \\ &= 4r_1 f_1 g_1 + (r_1 r_2 - z^2) f_2 g_2 + r_2 f_z g_z + 2z f_1 g_z + 2z f_z g_1\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(f, f) &= \hat{\Gamma}_2(f, f) \\ &= 16r_1^2 f_{11}^2 + 16r_1 f_1 f_{11} + 8r_1(r_1 r_2 - z^2) f_{12}^2 + 8(r_1 r_2 - z^2) f_2 f_{12} + 8(r_1 r_2 + z^2) f_{1z}^2 \\ &\quad + 32r_1 z f_{11} f_{1z} + r_1(r_1 r_2 - z^2) f_2 f_{22} + (r_1 r_2 - z^2)^2 f_{22}^2 + 2(r_1 r_2 - z^2) f_z f_{2z} \\ &\quad + 8z(r_1 r_2 - z^2) f_{12} f_{2z} + (2r_2 + \frac{r_1^2}{2}) f_2^2 + 2r_2(r_1 r_2 - z^2) f_{2z}^2 + r_2^2 f_{zz}^2 + 4r_2 f_1 f_{zz} \\ &\quad + 8r_2 z f_{1z} f_{zz} + 16z f_1 f_{1z} + 8z^2 f_{11} f_{zz} + 12f_1^2 + \frac{1}{2} r_1 f_z^2 - 4(r_1 r_2 - z^2) f_1 f_{22} \\ &\quad - 4r_1 f_1 f_2 - (r_1 r_2 - z^2) f_2 f_{zz} - 2z f_2 f_z.\end{aligned}$$

By careful study, we can express the above into a functional quadratic form.

$$\begin{aligned}\Gamma_2(f, f) &= \left((r_1 r_2 - z^2) f_{22} + \frac{r_1}{2} f_2 - 2f_1 \right)^2 + 8r_1(r_1 r_2 - z^2) \left(f_{12} + \frac{f_2}{2r_1} + \frac{z f_{2z}}{2r_1} \right)^2 \\ &\quad + \frac{2}{r_1} \left((r_1 r_2 - z^2) f_{2z} + \frac{r_1}{2} f_z - z f_2 \right)^2 + \left(\frac{r_1}{2} f_2 - 2f_1 - \frac{r_1 r_2 - z^2}{r_1} f_{zz} \right)^2 \\ &\quad + 4 \left(f_1 + \frac{z^2}{2r_1} f_{zz} + 2r_1 f_{11} + 2z f_{1z} \right)^2 + 2(r_1 r_2 - z^2) \left(\frac{z}{r_1} f_{zz} + 2f_{1z} \right)^2.\end{aligned}$$

Hence the desire result follows. \square

4 Gradient bounds for the heat kernels

As done in [3], we have the following Li-Yau type inequality holds.

Proposition 4.1. *There exist positive constants C_1, C_2, C_3 such that for any positive function f , if $u = \log P_t f$, we have*

$$\partial_t u \geq C_1 \Gamma(u) + C_2 t \sum_{i=1}^3 |Y_i u|^2 - \frac{C_3}{t}.$$

Proof. Here we briefly proof it for the readers' convenience. Notice that for all $\lambda > 0$,

$$\begin{aligned}\Gamma_2(f, f) &= \sum_{i=1}^3 (X_i^2 f)^2 + \frac{1}{2} \sum_{i=1}^3 (Y_i f)^2 + 2 \sum_{i=1}^3 (D_{i, i+1}(f))^2 + 2 \sum_{i=1}^3 (X_i f X_{i+2} Y_{i+1} f - X_{i+2} f X_i Y_{i+1} f) \\ &\geq \frac{1}{3} (\mathcal{L} f)^2 + \frac{1}{2} \sum_{i=1}^3 (Y_i f)^2 - 4\sqrt{\Gamma(f)} \cdot \sqrt{\sum_{i=1}^3 \Gamma(Y_i f)} \\ &\geq \frac{1}{3} (\mathcal{L} f)^2 + \frac{1}{2} \sum_{i=1}^3 (Y_i f)^2 - \frac{4}{\lambda} \Gamma(f) - \lambda \sum_{i=1}^3 \Gamma(Y_i f),\end{aligned}\tag{4.1}$$

where $D_{i,i+1} = \frac{1}{2}(X_i X_{i+1} + X_{i+1} X_i)$, and the last two inequalities follow from the Cauchy-Schwartz inequality.

Set $f_s = P_{t-s}f$, $u_s = \log f_s$, following [3], let

$$\Phi_1(s) = P_s(f_s \Gamma(u_s, u_s)), \quad \Phi_2(s) = P_s \left(f_s \sum_{i=1}^3 (Y_i u_s)^2 \right),$$

we have

$$\Phi'_1(s) = 2P_s(f_s \Gamma_2(u_s, u_s)), \quad \Phi'_2(s) = 2P_s \left(f_s \sum_{i=1}^3 \Gamma(Y_i u_s) \right).$$

Combining (4.1) and

$$(\mathcal{L}u_s)^2 \geq 2\gamma \mathcal{L}u_s - \gamma^2, \quad \mathcal{L}u_s = \frac{\mathcal{L}f_s}{f_s} - \Gamma(u_s),$$

we have

$$\Phi'_1(s) \geq \left(-\frac{4}{\lambda} - \frac{4\gamma}{3} \right) \Phi_1(s) + \Phi_2(s) - \lambda \Phi'_2(s) + \frac{4\gamma}{3} \mathcal{L}P_t f - \frac{2\gamma^2}{3} P_t f.$$

Denote a, b are positive functions defined on $[0, t)$, with b is decreasing, we have

$$(a(s)\Phi_1(s) + b(s)\Phi_2(s))' \geq \left(a' - \frac{4a}{\lambda} - \frac{4a\gamma}{3} \right) \Phi_1(s) + (a+b')\Phi_2(s) + (b-\lambda a)\Phi'_2 + \frac{4\gamma a}{3} \mathcal{L}P_t f - \frac{2\gamma^2 a}{3} P_t f.$$

By choosing

$$a = -b', \quad \lambda = -\frac{b}{b'}, \quad \gamma = \frac{3b''}{4b'} + \frac{3b'}{b},$$

and then choose $b(s) = (t-s)^\alpha$, for some $\alpha > 2$, integrating the above differential inequality from 0 to t , the desired result follows. \square

As a consequence, we have the following Harnack inequality: There exist positive constants A_1, A_2 , for $t_2 > t_1 > 0$, and $g_1, g_2 \in \mathfrak{N}_{3,2}$,

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1} \right)^{A_1} e^{A_2 \frac{d^2(g_1, g_2)}{t_2 - t_1}}. \quad (4.2)$$

Here is an analogue result of Theorem B in the three Brownian motions model.

Proposition 4.2. *There exists a constant $C > 0$ such that for $t > 0$, $g = (x, y) \in \mathfrak{N}_{3,2}$,*

$$\sqrt{\Gamma(\log p_t)(g)} \leq \frac{Cd(g)}{t},$$

where $p_t(g)$ denotes the density of P_t at 0 and $d(g)$ denotes the Carnot-Carathéodory distance between 0 and g .

Proof. Following [4], for $0 < s < t$, let $\Phi(s) = P_s(p_{t-s} \log p_{t-s})$, we have

$$\Phi'(s) = P_s(p_{t-s} \Gamma(\log p_{t-s})), \quad \Phi''(s) = 2P_s(p_{t-s} \Gamma_2(\log p_{t-s})).$$

By Proposition 3.1, Φ'' is positive, whence Φ' is non-decreasing, thus

$$\int_0^{\frac{t}{2}} \Phi'(s) ds \geq \frac{t}{2} \Phi'(0).$$

That is

$$p_t \Gamma(\log p_t) \leq \frac{2}{t} (P_{t/2}(p_{t/2} \log p_{t/2}) - p_t \log p_t).$$

The right hand side can be bounded by applying the above Harnack inequality (4.2) and the basic fact $p_{t/2}(g) \leq p_{t/2}(0)$, for all $g \in \mathfrak{N}_{3,2}$. We have

$$\sqrt{\Gamma(\log p_t)(g)} \leq C \left(\frac{d(g)}{t} + \frac{1}{\sqrt{t}} \right).$$

In particular,

$$\sqrt{\Gamma(\log p_1)(g)} \leq C (d(g) + 1).$$

If $d(g) \geq 1$, it is trivial to get the desired result.

Note that for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, see P. 125, Theorem 1 in [8],

$$\begin{aligned} p_1(x, y) &= -(2\pi)^{-\frac{15}{2}} \int_{\mathbb{R}^3} \exp(-iy \cdot \alpha) \frac{|\alpha|}{2} \left(\sinh \frac{|\alpha|}{2} \right)^{-1} \\ &\quad \cdot \exp \frac{1}{2} \left\{ -|x|^2 - \frac{x A^2 x^t}{|\alpha|^2} \left(1 - \frac{|\alpha|}{2} \coth \frac{|\alpha|}{2} \right) \right\} \prod_{k=1}^3 d\alpha_k \\ &\stackrel{(*)}{=} -(2\pi)^{-\frac{15}{2}} \int_{\mathbb{R}^3} \exp(-iy \cdot \alpha) \frac{|\alpha|}{2} \left(\sinh \frac{|\alpha|}{2} \right)^{-1} \\ &\quad \cdot \exp \frac{1}{2} \left\{ -|x|^2 + (|x|^2 - (x \cdot \tilde{\alpha})^2) \left(1 - \frac{|\alpha|}{2} \coth \frac{|\alpha|}{2} \right) \right\} \prod_{k=1}^3 d\alpha_k, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha|^2 = \sum_{k=1}^3 \alpha_k^2$, $\tilde{\alpha} = \frac{1}{|\alpha|}(\alpha_1, \alpha_2, \alpha_3)$ and $A = \begin{pmatrix} 0 & \alpha_1 & -\alpha_2 \\ -\alpha_1 & 0 & \alpha_3 \\ \alpha_2 & -\alpha_3 & 0 \end{pmatrix}$. (*)

follows from $x A^2 x^t = -|\alpha|^2(|x|^2 - (x \cdot \alpha)^2)$. Notice

$$\begin{aligned} \partial_j p_1(x, y) &= (2\pi)^{-\frac{15}{2}} \int_{\mathbb{R}^3} \exp(-iy \cdot \alpha) |\alpha| \left(\sinh \frac{|\alpha|}{2} \right)^{-1} \cdot \left(-x_j + (x_j - x \cdot \tilde{\alpha} \tilde{\alpha}_j) \left(1 - \frac{|\alpha|}{2} \coth \frac{|\alpha|}{2} \right) \right) \\ &\quad \cdot \exp \frac{1}{2} \left\{ -|x|^2 + (|x|^2 - (x \cdot \tilde{\alpha})^2) \left(1 - \frac{|\alpha|}{2} \coth \frac{|\alpha|}{2} \right) \right\} \prod_{k=1}^3 d\alpha_k \end{aligned}$$

and

$$\begin{aligned} \hat{\partial}_i p_1(x, y) &= (2\pi)^{-\frac{15}{2}} i \int_{\mathbb{R}^3} \exp(-iy \cdot \alpha) \frac{|\alpha|}{2} \left(\sinh \frac{|\alpha|}{2} \right)^{-1} \cdot \alpha_i \\ &\quad \cdot \exp \frac{1}{2} \left\{ -|x|^2 + (|x|^2 - (x \cdot \tilde{\alpha})^2) \left(1 - \frac{|\alpha|}{2} \coth \frac{|\alpha|}{2} \right) \right\} \prod_{k=1}^3 d\alpha_k. \end{aligned}$$

Let

$$W_1 = \int_{\mathbb{R}^3} |\alpha| \left(\sinh \frac{|\alpha|}{2} \right)^{-1} \prod_{k=1}^3 d\alpha_k, \quad W_2 = \int_{\mathbb{R}^3} \left(\sinh \frac{|\alpha|}{2} \right)^{-1} \cdot |\alpha|^2 \coth \frac{|\alpha|}{2} \prod_{k=1}^3 d\alpha_k,$$

obviously we have W_1, W_2 are bounded. For $g = (x, y) \in \mathfrak{N}_{3,2}$, satisfying $d(g) \leq 1$, by the basic fact that $|x| \leq d(g) \leq 1$ (see [16, 17], in fact we can easily proof it on the nilpotent groups.), we have

$$\sqrt{\Gamma(p)(g)} \leq C_1|x|(W_1 + W_2) \leq C_2|x| \leq Cd(g).$$

The desired result follows by the time scaling property (1.7). □

Notice that for any radial function f , $P_t f$ is also radial since all θ_i commute with P_t . Thanks to Proposition 3.1, we have H. Q. Li inequality, LSI inequality, isoperimetric inequalities etc. hold for the semigroup restricted on the radial functions, see [2, 1]. We state them in the following proposition.

Proposition 4.3. *For any compactly supported smooth, radial function f , for any $t \geq 0$, $g \in \mathfrak{N}_{3,2}$,*

- (i) *H. Q. Li inequality.* $\Gamma(P_t f, P_t f)^{\frac{1}{2}}(g) \leq P_t(\Gamma(f, f)^{\frac{1}{2}})(g).$
- (ii) *LSI inequality.* $P_t(f \log f)(g) - P_t(f) \log P_t(f)(g) \leq t P_t\left(\frac{\Gamma(f, f)}{f}\right)(g).$
- (iii) *Isoperimetric inequality.* $P_t(|f - P_t(f)(g)|)(g) \leq 4\sqrt{t} P_t(\Gamma(f)^{\frac{1}{2}})(g).$

Discussion: Here we have shown that H. Q. Li inequality holds for the radial functions. For the general functions, it is still open. Following the viewpoint of [11], also [2], one key point to proof H. Q. Li inequality is the precise lower and upper bounds for the associated heat kernel. But in our setting, this estimates is unknown, at least the methods in [11]-[13] are not applicable. This precise estimates are also essential to proof the cheeger type inequality, see Lemma 5.1 in [2] and proof the constant coefficient is bounded in the complex quasi-communication method, see Proposition 5.5 in [2].

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